# MA2104 Multivariable Calculus

## Basic Vectors

- **Thm 1**: $\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$
- **Thm 2**: (unit vector in direction of $\mathbf{a}$) $= \frac{\mathbf{a}}{\|\mathbf{a}\|}$
- **Thm 3 [Dot product properties]**:

$$\begin{array}{l|l} \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} & (d\mathbf{a})\cdot\mathbf{b} = d\,(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(d\mathbf{b}) \\ \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} & \mathbf{0}\cdot\mathbf{a} = 0 \\ (\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|^2 \end{array}$$

- **Thm 4 [Dot product & angle]**: $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$
- **Thm 5 [Orthogonality]**: $\mathbf{a}\perp\mathbf{b} \iff \mathbf{a}\cdot\mathbf{b} = 0$
- **Component (signed scalar)**: $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$
- **Projection (vector)**: $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \operatorname{comp}_{\mathbf{a}}\mathbf{b} \times \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$
- **Cross product**: $\langle a_1,a_2,a_3\rangle \times \langle b_1,b_2,b_3\rangle := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$
  $\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$
- **Thm 6**: $(\mathbf{a}\times\mathbf{b})\perp\mathbf{a}$ and $(\mathbf{a}\times\mathbf{b})\perp\mathbf{b}$
- **Thm 7 [Cross prod. & angle]**: $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta$
- **Thm 8 [Cross product properties]**:

$$\begin{array}{l|l} \mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a}) & (\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c} \\ \mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} & (d\mathbf{a})\times\mathbf{b} = d\,(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(d\mathbf{b}) \end{array}$$

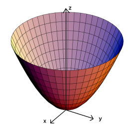
- **Scalar triple product** (= signed vol. of parallelepiped):
  $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) := \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$
  $a_1(b_2c_3 - b_3c_2), a_2(b_3c_1 - b_1c_3), a_3(b_1c_2 - b_2c_1)$
- **Thm 10 & 11 [Plane]**:
  $\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{r_0} \iff ax+by+cz = ax_0+by_0+cz_0 = d$
- **Thm 13 [Derivative properties for vectors]**:
  $\frac{d}{dt}(\mathbf{r}(t)+\mathbf{s}(t)) = \mathbf{r}'(t)+\mathbf{s}'(t)$
  $\frac{d}{dt}(c\,\mathbf{r}(t)) = c\,\mathbf{r}'(t)$
  $\frac{d}{dt}(f(t)\,\mathbf{r}(t)) = f'(t)\,\mathbf{r}(t)+f(t)\,\mathbf{r}'(t)$
  $\frac{d}{dt}(\mathbf{r}(t)\cdot\mathbf{s}(t)) = \mathbf{r}'(t)\cdot\mathbf{s}(t)+\mathbf{r}(t)\cdot\mathbf{s}'(t)$
  $\frac{d}{dt}(\mathbf{r}(t)\times\mathbf{s}(t)) = \mathbf{r}'(t)\times\mathbf{s}(t)+\mathbf{r}(t)\times\mathbf{s}'(t)$
- **Thm 14 [Arc length]**: (length from $a$ to $b$) $= \int_a^b \|\mathbf{r}'(t)\|\,dt$
- **Vector rotation**: 90° anticlockwise: $\langle x,y\rangle \to \langle -y,x\rangle$
  90° clockwise: $\langle x,y\rangle \to \langle y,-x\rangle$

## Surfaces

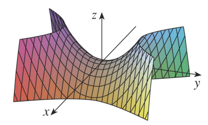
- **Level curve** of $f(x,y)$ = horizontal trace (for functions in two vars) = 2-D graph of $f(x,y)=k$ for some constant $k$
  **Contour plot** = numerous level curves on the same graph
- **Level surface** of $f(x,y,z)$ = 3-D graph of $f(x,y,z)=k$ for some constant $k$.

Quadric surfaces

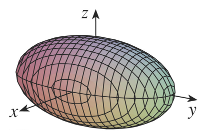
- **Cylinder** = infinite prism
- **Elliptic paraboloid**: $\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{z}{c}$



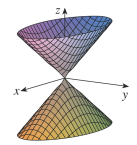
- **Hyperbolic paraboloid**: $\frac{x^2}{a^2}-\frac{y^2}{b^2} = \frac{z}{c}$



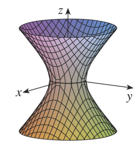
- **Ellipsoid**: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$



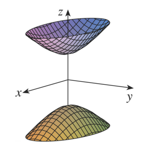
- **Elliptic cone**: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 0$



- **Hyperboloid of one sheet**: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1$



- **Hyperboloid of two sheets**: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = -1$



## Limits

- **Limit**: $\lim_{(x,y)\to(a,b)} f(x,y) = L$
  iff for any $\epsilon > 0$ there exists $\delta > 0$ such that
  $|f(x,y)-L| < \epsilon$ whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$
- **Thm 15**: To show limit does not exist, take the limit via two different paths that have different limits
- **Thm 16 & 17 [Limit theorems]**: Limits may be taken into addition, subtraction, multiplication, division
- **Thm 18 [Squeeze theorem]**:
  If $|f(x,y)-L| \le g(x,y)\ \forall(x,y)$ close to $(a,b)$
  and $\lim_{(x,y)\to(a,b)} g(x,y) = 0$
  then $\lim_{(x,y)\to(a,b)} f(x,y) = L$
- **Continuity**: $f$ is continuous at $(a,b)$
  $\iff \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
  i.e. the limit exists and the $f$ is valid at $(a,b)$
- **Thm 20 & 21 [Continuity theorems]**:
  If two functions are continuous (at $(a,b)$), then their sum, difference, product, quotient, and composition are continuous too (quotient requires denominator $\neq 0$)
- All polynomials, trigonometric, exponential, and rational functions are continuous

## Partial Derivatives

- **Thm 2 [Clairaut's theorem]**: If $f_{xy}$ and $f_{yx}$ are both continuous on disk containing $(a,b)$ then $f_{xy}(a,b) = f_{yx}(a,b)$
- **Thm 3 [Tangent plane eqn]**:
  Given surface $z = f(x,y)$ with point $(a,b)$:
  - normal vector: $\langle f_x(a,b), f_y(a,b), -1\rangle$
  - tangent plane: $z = f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$
- **Multivariable differentiability**:
  $z = f(x,y)$ is differentiable at $(a,b)$ if
  $\triangle z = f_x(a,b)\triangle x + f_y(a,b)\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y$
  (with vanishing $\epsilon_1$ and $\epsilon_2$) *i.e. zooming in to $(a,b)$ will make surface approximate tangent plane*
- $f_x$ & $f_y$ are continuous at $(a,b) \implies f$ is diff.able at $(a,b)$
- $f$ is differentiable at $(a,b) \implies f$ is continuous at $(a,b)$

## Differentiation Techniques

- **Chain rule**: For $z = f(x,y)$ and $x = x(t)$, $y = y(t)$:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

  For $z = f(x,y)$ and $x = x(s,t)$, $y = y(s,t)$:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

- **Thm 11 [Implicit differentiation]**: Given $F(x,y,z) = 0$ that defines $z$ implicitly as a function of $x$ and $y$, then:

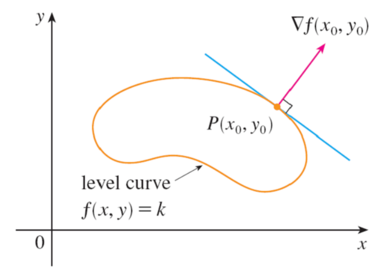
$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

  provided $F_z(x,y,z) \neq 0$
- **Quotient rule**:
  $f(x) = \frac{g(x)}{h(x)} \implies f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$
- $F(x,y,z) = 0 \implies$ normal vector $= \langle F_x, F_y, F_z\rangle$

## Gradient Vectors

- **Thm 13 [Dir. derivatives]**: $D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot\mathbf{u}$
  where $\nabla f(x,y) := \langle f_x, f_y\rangle$ = gradient vector at $(x,y)$
  and $\mathbf{u}$ := direction (as unit vector)
- Direction of $\nabla f(x,y)$ = steepest upward direction
  $\|\nabla f(x,y)\|$ = steepest upward gradient
- **Thm 1 [Level curve $\perp \nabla f$]**: $\mathbf{0} \neq \nabla f(x_0,y_0)$ is normal to the level curve $f(x,y) = k$ that contains $(x_0,y_0)$




- **Thm 2 [Level surface $\perp \nabla F$]**: $\mathbf{0} \neq \nabla F(x_0,y_0,z_0)$ is normal to the level surface $F(x,y,z) = k$ that contains $(x_0,y_0,z_0)$

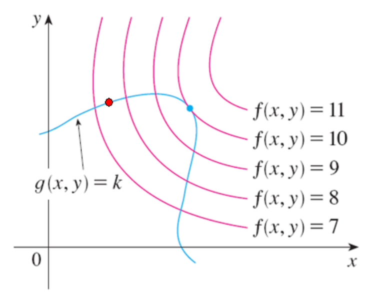
## Critical Points, Minimum, Maximum

given $f(x,y)\colon D \to \mathbb{R}$

- **Local maximum**: $(a,b)$ is a local maximum if $f(x,y) \le f(a,b)$ for all points $(x,y)$ near $(a,b)$
- **Local minimum**: $(a,b)$ is a local minimum if $f(x,y) \ge f(a,b)$ for all points $(x,y)$ near $(a,b)$
- **Saddle point**: $(a,b)$ is a saddle point if $f_x(a,b) = f_y(a,b) = 0$ and every neighbourhood at $(a,b)$ contains points $(x,y)\in D$ for which $f(x,y) < f(a,b)$ and points $(x,y)\in D$ for which $f(x,y) > f(a,b)$
- **Critical point**: $(a,b)$ is a critical point if $f_x(a,b) = f_y(a,b) = 0$
  *(If point $P$ is a local maximum/minimum then: $f_x(P)$ and $f_y(P)$ both exist $\implies$ $P$ is a critical point)*
- *Local maximum/minimum and critical points cannot be boundary points*
- **Absolute maximum**: $f$ has an absolute max. at $(a,b)$ if $\forall(x,y)\in D, f(x,y) \le f(a,b)$
- **Absolute minimum**: $f$ has an absolute min. at $(a,b)$ if $\forall(x,y)\in D, f(x,y) \ge f(a,b)$
- **Boundary point** of $R$: point $(a,b)$ such that every disk with center $(a,b)$ both contains points in $R$ and not in $R$
- **Closed set**: Set that contains all its boundary points
- **Bounded set**: Set that is contained in some (finite) disk
- **Thm 14 [Extreme Value Theorem]**:
  If $f(x,y)$ is continuous on a closed & bounded set $D$, then the absolute maximum & minimum must exist
- To find absolute maximum/minimum of $f$ with domain $D$:
  1) Find the values of $f$ at all critical points in $D$
  2) Find the extreme values of $f$ on the boundary of $D$
  3) Take largest/smallest of the values of Steps 1 & 2

## Lagrange Multipliers

- Suppose $f(x,y)$ and $g(x,y)$ are differentiable functions such that $\nabla g(x,y) \neq \mathbf{0}$ on the constraint curve $g(x,y) = k$.
  If $(x_0,y_0)$ is a (local) maximum/minimum of $f(x,y)$ constrained by $g(x,y) = k$, then $\nabla f(x_0,y_0) = \lambda\nabla g(x_0,y_0)$ for some constant $\lambda$ (the Lagrange multiplier).




- To find the maximum/minimum points of $f(x,y)$ constrained by $g(x,y) = k$, we solve

$$\begin{cases} \nabla f(x_0,y_0) = \lambda\nabla g(x_0,y_0) \\ g(x_0,y_0) = k \end{cases}$$

  for $x_0$, $y_0$, $\lambda$.

# Integration Techniques

- **Integration by parts**:

$$\int u\,\frac{dv}{dx}\,dx = u\,v - \int \frac{du}{dx}\,v\,dx$$

# Area & Volume Integrals

- **Thm 4 [Fubini's theorem]**:
  If $f$ is continuous on rectangle $R = [a,b] \times [c,d]$ then:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

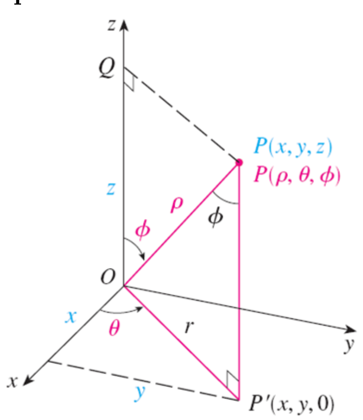
- **Region types (double integration)**:
  Type I: $D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$
  Type II: $D = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$
- **Polar coords. $\longleftrightarrow$ rectangular coords.**:

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta \qquad \theta = \operatorname{atan2}(y,x)$$

- **Integrating over a polar rectangle**:
  If $R = \{(r,\theta) : 0 \le a \le r \le b,\ \alpha \le \theta \le \beta\}$ then:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

- **Region types (polar)**:
  Type I: $D = \{(r,\theta) : 0 \le a \le r \le b,\ g_1(r) \le \theta \le g_2(r)\}$
  Type II: $D = \{(r,\theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$
- **Region types (triple integration)**:
  Type I: $E = \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$
  Type II: $E = \{(x,y,z) : (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$
  Type III: $E = \{(x,y,z) : (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$
- **Spherical coords. $\longleftrightarrow$ rectangular coords.**:




$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

$$\rho = \sqrt{x^2+y^2+z^2}$$
$$\theta = \operatorname{atan2}(y,x)$$
$$\phi = \cos^{-1}\frac{z}{\rho} = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

- **Integrating over a spherical wedge**:
  If $E = \{(\rho,\theta,\phi) : 0 \le a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$ then:

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

- **Jacobian (2D)** of transformation $(u,v) \mapsto (x,y)$:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

- **Jacobian (3D)** of transformation $(u,v,w) \mapsto (x,y,z)$:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

- Consider choosing transformation to make bounds constants
- Using $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$ and appropriate $f(x,y)$ may avoid needing to express $x, y$ in terms of $u, v$

# Line Integrals

- **Line integral for scalar field**:
  If curve $C$ is given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$ then:

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\,\|\mathbf{r}'(t)\|\,dt$$

  Answer is indep. of orientation and parameterization of $\mathbf{r}(t)$
- **Line integral for vector field**:
  If curve $C$ is given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$ then:

$$\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t))\cdot\mathbf{r}'(t)\,dt =$$
$$\int_C (P\,dx + Q\,dy + R\,dz) = \int_a^b Px'(t)dt + \int_a^b Qy'(t)dt + \int_a^b Rz'(t)dt$$

  Answer is its negation when $\mathbf{r}(t)$ has opposite orientation

## Conservative vector fields

- A vector field $\mathbf{F}$ is conservative on $D$ iff $\underline{\mathbf{F} = \nabla f}$ for some scalar function $f$ on $D$
  $f$ is called the <u>potential function</u> of $\mathbf{F}$
- To recover $f$ from $\mathbf{F} = \langle f_x, f_y\rangle$, do partial integration of $f_x$ to get $g(x,y) + h(y)$ $[= f(x,y)]$ (where $h(y)$ is the unknown integration constant), then differentiate $g(x,y) + h(y)$ w.r.t. $y$ and compare with $f_y$ to determine $h(y)$
- **Test for conservative field (2D)**:
  If $\mathbf{F} = \langle P, Q\rangle$ is a vector field in an <u>open</u> *(excludes all boundary points)* and <u>simply-connected</u> *(has no "holes")* region $D$ and both $P$ and $Q$ have <u>continuous first-order partial derivatives</u> on $D$ then:

$$\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y} \iff \mathbf{F} \text{ is conservative on } D$$

- **Test for conservative field (3D)**:
  $\mathbf{F} = \langle P, Q, R\rangle$ *(similar requirements as 2D case)*:

$$\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y},\ \frac{\partial R}{\partial y} \equiv \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} \equiv \frac{\partial R}{\partial x} \iff \mathbf{F} \text{ is conservative on } D$$

- **Fundamental theorem for line integrals**:
  If $\mathbf{F}$ is conservative with potential function $f$, and $C$ is a smooth curve from point $A$ to point $B$, then:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$$

  $\Longrightarrow$ line integral for conservative field is path-independent
- Two paths with different line integrals but same initial and terminal points $\Longrightarrow$ vector field is <u>not</u> conservative

# Green's Theorem

- If $C$ is a positively oriented (anticlockwise), piecewise smooth, simple closed curve in the plane, and $D$ is the region bounded by $C$, and $\mathbf{F} = \langle P, Q\rangle$ then:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

  (useful when $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is simpler than $P$ and $Q$)
- **!!!** *<u>Look out for holes</u>* $(\div 0)$ – use <u>extended</u> Green's theorem
- **!!!** When borrowing other question result, check <u>orientation</u>
- **Reverse application of Green's theorem**:
  If $A$ is the area of $D$, then *(choose whichever is convenient)*:

$$A = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

  Parameterize the boundary curve in terms of $t$ $(a \le t \le b)$
  (e.g. $\frac{1}{2}\int_a^b \left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt}\right) dt$ )

# Surface Integrals

- **Parametric form of a surface in $\mathbb{R}^3$**:
  $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $\quad (u,v) \in D$
- **Smooth surface**: A surface that is parameterized by $\mathbf{r}(u,v)$ where $(u,v) \in D$, such that $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v \ne \mathbf{0}\ \forall\ (u,v) \in D$
- **Thm 6 [Normal vector of parametric surface]**:
  If a smooth surface $S$ has parameterization $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$ then:
  $\mathbf{r}_u(a,b) \times \mathbf{r}_v(a,b)$ is normal to $S$ at $(x(a,b), y(a,b), z(a,b))$
- **Thm 7 [Surface integral for scalar field]**:
  If a smooth surface $S$ has parameterization $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$ then:

$$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v), y(u,v), z(u,v))\,\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA$$

- **Thm 7a [Surface integral special case $z = g(x,y)$]**:
  If $S$ is the surface $z = g(x,y)$ where $(x,y) \in D$ then:

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\right) dA$$

- **Orientable surface**: two-sided surface
  **Positive orientation**: outward from enclosed region
- **Thm 6 [Surface integral for vector field]**:
  If a smooth surface $S$ has parameterization $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$ then:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$$

- **Thm 6a [Surface integral special case $z = g(x,y)$]**:
  If $\mathbf{F} = \langle P, Q, R\rangle$, and $S$ is the surface $z = g(x,y)$ where $(x,y) \in D$, then the flux in the <u>upward</u> orientation:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

# Vector Differential Operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle$$

# Divergence

If $\mathbf{F} = \langle P, Q, R\rangle$ then:

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

## Gauss' theorem

If $E$ is a solid region with piecewise smooth boundary surface $S$ with positive (outward) orientation then:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV$$

# Curl

If $\mathbf{F} = \langle P, Q, R\rangle$ then:

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle$$

## Stokes' theorem

If $S$ is a surface with a boundary curve $C$ (positively oriented w.r.t. $S$) then:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

**Positive orientation of boundary curve**:
If surface $S$ has unit normals pointing towards you, then the positive orientation of boundary curve $C$ goes anti-clockwise

# Trigonometric Formulae

<u>Double angle</u>

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

<u>Triple angle</u>

$$\sin 3x = 3\sin x - 4\sin^3 x$$
$$\cos 3x = 4\cos^3 x - 3\cos x$$

<u>Pythagorean</u>

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\cot^2 x + 1 = \csc^2 x$$

<u>Integrals</u>

$$\int \sin^2 x\,dx = \frac{1}{4}(2x - \sin 2x)$$
$$\int \cos^2 x\,dx = \frac{1}{4}(2x + \sin 2x)$$
$$\int \tan^2 x\,dx = \tan x - x$$
$$\int \sin^3 x\,dx = \frac{1}{12}(\cos 3x - 9\cos x)$$
$$\int \cos^3 x\,dx = \frac{1}{12}(\sin 3x + 9\sin x)$$
$$\int \sin x\cos x\,dx = -\frac{1}{2}\cos^2 x + C_1 = \frac{1}{2}\sin^2 x + C_2$$

<u>Sum of angles</u>

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$
$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$$